Caratheodory Convergence.

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Geometric characterization of convergence in (S).

Def. Let Ω_n - sequence of s.c. domains. $w_0 \in \Omega_n \forall n$.
The kernel of Ω_n with respect to w_0 is:
$w_0 \cup \{w \in \mathbb{C} : \exists$ domain $H : w_0 \in H, w \in H, \exists N: H \subset \Omega_n \ \forall n \geq N\}$.

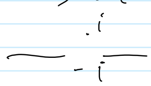
No such w - kernel is $\{w_0\}$.

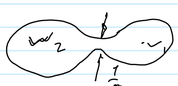
Otherwise: $\ker(\{\Omega_n\}, w_0)$ is a domain, $=: \Omega$. $w_0 \in \Omega$.

Def. $(\Omega_n, w_0) \to (\Omega, w_0)$ if $\forall n_k$ - subsequence, $\ker(\Omega_{n_k}, w_0) = \Omega$.

Example. Let $\Omega_{n+1} \subset \Omega_n$. $\Omega :=$ component of int $(\cap \Omega_n)$ a priori, smaller containing w_0.
Then $\Omega_n \xrightarrow{\ker} \Omega$ if $\Omega \neq \emptyset$, and $\Omega_n \xrightarrow{\ker} \{w_0\}$ otherwise!

Pf By monotonicity, $\ker(\Omega_{n_k}, w_0) = \ker(\Omega_n, w_0)$. So need: $\Omega = \ker$.
$\Omega \subset \ker$ (since can take $H = \Omega$).
On the other hand: $w \in \ker \Rightarrow \exists H \subset \Omega_n \ \forall n$ (monotonicity!) $\Rightarrow H \subset \cap \Omega_n$.
so H - open, $w_0 \in H$ $\Rightarrow H \subset \Omega \Rightarrow w \in \Omega$ #

Examples (to example). 1)  $\Omega_n = \mathbb{C} \setminus \{(-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, \infty)\}$.

2)  Ω_n

Equivalent def. $\Omega_n \xrightarrow{\ker} \Omega$, $w_0 \in \Omega_n$, $w_0 \in \Omega$ iff
1) $\forall K$ - compact $K \subset \Omega \Rightarrow \exists N: K \subset \Omega_n \ \forall n > N$.
2) $\forall c \in \partial\Omega \ \exists c_n \in \partial\Omega_n : c_n \to c$. (2 can be restated $\operatorname{dist}(c, \partial\Omega_n) \to 0$).

Pf (of equivalency).
(I). Let $\Omega_n \xrightarrow{\ker} \Omega$. $K \subset \Omega$ - compact. $\forall x \in K \ \exists H_x$ - domain, $x \in H_x$,
ker $\Rightarrow$ $w_0 \in H_x$. $\cup H_x \supset K \underset{\text{compact}}{\Rightarrow} K \subset \bigcup_{l=1}^{M} H_l \subset \Omega_n$ for large enough n. $\Rightarrow$
$\Omega \subset \ker \Omega_n$.
Take now $c \in \partial\Omega$. If $\nexists c_n$, then $\exists \varepsilon > 0, n_k \uparrow \infty : B(c, \varepsilon) \cap \partial\Omega_{n_k} = \emptyset$.
Take $w \in B(c, \varepsilon) \cap \Omega$. $\cancel{\Omega} = \ker \Rightarrow w \in H \subset \Omega_n$ for large n,
so $H \cup B(c, \varepsilon) \subset \Omega_{n_k}$ for large $k \Rightarrow c \in \ker(\Omega_{n_k})$ - contradiction.

(II) $\Rightarrow$ ker. $w \in \Omega \Rightarrow \exists K \subset \Omega : \{w, w_0\} \subset \operatorname{Int} K$, Int K - connected.
Then $K \subset \Omega_n$ for large n. so $\operatorname{Int} K \subset \Omega_n$, take $H = \operatorname{Int} K \Rightarrow K \subset \ker \Omega_n$.
On the other hand, $c \in \partial\Omega \Rightarrow c \notin \ker(\Omega_{n_k})$ ($c_{n_k} \to c$,
$H \ni c \Rightarrow c_{n_k} \in H$ for large k - contradiction with $H \subset \Omega_{n_k}$).
so $\partial\Omega \subset \mathbb{C} \setminus \ker(\Omega_{n_k}) \ \forall (n_k) \Rightarrow \Omega = \ker(\Omega_{n_k})$ ∎

Yet another equiv. def. (conv. in the sense of interior approximation).
Connected compact $K \subset \Omega_1 \cap \Omega_2$ is called common ε-interior approximation if
1) $w_0 \in K$
2) $\forall w \in \partial K \ \exists w_1 \in \partial\Omega_1, w_2 \in \partial\Omega_2 : |w - w_1| < \varepsilon, |w - w_2| < \varepsilon$.

$\Omega_n \xrightarrow{\text{int. approx}} \Omega$ if $\exists \varepsilon > 0 \ \exists K \subset \Omega$ and N: K is common ε-int. approx. for Ω_n and Ω $\forall^{w_0 \in} n \geq N$.

Pf. (ker $\Rightarrow$ Int). Take K_ε - an $\frac{\varepsilon}{2}$ interior approx. to Ω. K_ε - compact $\Rightarrow$
$K_\varepsilon \subset \Omega_n \ \forall n \geq N$. Take $z \in \partial K_\varepsilon$. If $\exists n_k \uparrow \infty : \operatorname{dist}(z, \partial\Omega_{n_k}) > \varepsilon$, then
$c \in \partial\Omega : |c - z| < \frac{\varepsilon}{2}$.

$\mathrm{dist}(K, \partial\Omega_n) > \frac{\varepsilon}{2}$ + contradiction

($\ker \Rightarrow \ker$). $F \subset \Omega$ - compact. Take $\varepsilon < \mathrm{dist}(F, \partial\Omega)$.
Then $F_1 \subset K_\varepsilon$, so $K \subset \Omega_n$, $n \geq N$. Take $F_1 \supset F \cup \{w_0\}$ - connected compact. Thus $\Omega \subset \ker(\Omega_n)$.
If $w \in \ker(\Omega_{n_k}) \setminus \Omega$, then $\exists\, \mathcal{H} \subset \cap\Omega_{n_k}$, $\mathcal{H} \ni w, w_0$ - open. $\Rightarrow$
. Observe: $\Omega = \cup K_\varepsilon$. Then $\mathcal{H} \cup \cup K_\varepsilon \supset \Omega$, $\Omega_{n_k} \supset \mathcal{H} \Rightarrow$
$\forall z \in \partial K_\varepsilon$: $\mathrm{dist}(z, \partial\mathcal{H}) < \varepsilon$. Take $w \in \mathcal{H} \setminus \Omega$. $\exists F$ - connected compact in $\mathcal{H}$,
$w, w_0 \in F$. Let $2\varepsilon = \mathrm{dist}(F, \partial\mathcal{H})$. Then any ε-int. approx. of $\mathcal{H} \cup \Omega$ containing
F. Thus $K_\varepsilon \ni w$ - contradiction ∎

Thm (Carathéodory). Let $f_n: \mathbb{D} \to \Omega_n$ - univalent, $f_n(0) = w_0$, $f_n'(0) > 0$.
$f: \mathbb{D} \to \Omega$, $f(0) = w_0$, $f'(0) > 0$.
Then $f_n(z)$ converges locally uniformly to $f \iff \Omega_n \xrightarrow{\ker} \Omega$, $\Omega \neq \mathbb{C}$.

Pf (I) $f_n \to f \Rightarrow \Omega = \ker(\Omega_n)$. (since also $f_{n_k} \to f$, $\Omega = \ker(\Omega_{n_k})$, so
it is enough to prove this direction. Also, suppose $f \not\equiv$ const, f is univalent, so
$\Omega = f(\mathbb{D}) \neq \mathbb{C}$.
Let $w \in \Omega$. Let us prove $w \in \ker(\Omega_n)$. If $f \equiv w_0$ nothing to
prove. Let $w = f(z)$. Take r: $|z| < r < 1$. $\mathcal{H} := \{f(z) : |z| < r\}$.
Need: $\mathcal{H} \subset \Omega_n$ for large n. Assume not. $\exists n_k \to \infty$, $w_k \in \mathcal{H}$: $w_k \notin \Omega_{n_k}$.
By passing to a subsequence, assume $w_{n_k} \to w^*$. Rouché!
$\forall z \in \mathbb{D}$: $f_{n_k}(z) - w_k = (f_{n_k}(z) - w_k + w^*) - w^* \neq 0 \Rightarrow f(z) - w^* \neq 0$.
But $w^* \in \overline{\mathcal{H}} \subset \Omega$ - contradiction.
Let $w \in \ker(\Omega_n)$. Take $\mathcal{H} \ni \{w, w_0\}$. For large n, $\mathcal{H} \subset \Omega_n$, so
$g_n (= f_n^{-1})$ - univalent on $\mathcal{H}$, $g_n(w_0) = 0$, $g_n'(0) > 0$. $g_n(\mathcal{H}) \subset \mathbb{D}$. By Montel, $\exists$
$g_{n_k} \to g$. $g(0) = 0$, $|g(w)| \leq 1$ in $\mathcal{H}$. so, by Maximum, $g(\mathcal{H}) \subset \mathbb{D}$.
f_{n_k} converges locally uniformly near $z := g(w)$. Then $g_{n_k}(w) \to g(w)$ and
$f_{n_k}(g_{n_k}(w)) = w$, we get $w = f(z)$ so $w \in \Omega$ ∎

(II) Let $\Omega_n \xrightarrow{\ker} \Omega$, $\Omega \neq \mathbb{C}$.
First, let us show that f_n is normal.
Know: $\{|w - w_0| < \frac{1}{4} f_n'(0)\} \subset \Omega_n$. $f_n'(0)$ - unbounded $\Rightarrow \exists (\Omega_{n_k})$:
(by Köbe)
$\mathbb{C}$ is kernel of Ω_{n_k}. so $|f_n'(0)|$ - bounded.
Also, $|f_n(z)| \leq |f_n'(0)| \frac{|z|}{(1-|z|)^2}$, so $f_n(z)$ is locally bounded $\Rightarrow$ normal.
(since $\frac{f_n(z)}{f_n'(0)} \in S$)
Assume $f_n \not\to f$. Then, by normality, $\exists f_{n_k} \to f^* \neq f$.
By (I), $\Omega_{n_k} \to \Omega^* = f^*(\Omega)$. By $\Omega_{n_k} \to \Omega$ - contradiction! ∎